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Identical synchronization, with translation invariance, implies parameter estimation

Gregory S. Duane [a,*], Dongchuan Yu [b], Ljupco Kocarev [c]

[a] National Center for Atmospheric Research, PO Box 3000, Boulder, CO 80307, USA
[b] College of Automation Engineering, Qingdao University, 308 Ningxia Road, Qingdao, Shandong 266071, People's Republic of China
[c] Institute for Nonlinear Science, University of California, San Diego 9500 Gilman Drive, La Jolla, CA 92093-0402, USA

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Abstract

If a real system and its perfect model (with identical parameters) can be made to synchronize identically, then for an imperfect model of a real system with some unknown parameters, one can design a parameter estimation law, so that all parameters of the real system can be estimated from the model. While the information needed to implement the law is not available in all cases, sufficient information is generally available for a PDE system that possesses translational symmetry if some of the relevant state variables are known at the discrete set of points (or for the finite set of Fourier components) that are coupled to the model. Parameter estimation is illustrated for a geophysical fluid dynamics model.

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The synchronization of loosely coupled chaotic systems extends the older idea of synchronized periodic oscillators in nature [1]. Synchronized chaos has been described in electronic circuits [2], partial differential equations [3], and, recently, atmospheric models [4]. In many practical applications, one is concerned with the *synthesis* problem in synchronization—how to design a control signal or coupling scheme for two chaotic systems so that they are guaranteed to synchronize. If one system represents "reality" and the other a "computational model", such a coupling can be used to effect *data assimilation* [6] from a stream of noisy measurements into a running model that will effectively predict the future of the true system [7,8]. In data assimilation with an imperfect model, and more generally in a Dynamic Data Driven Applications System (DDDAS) [9] that dynamically integrates experimental measurements to adapt observational strategy, one seeks to change model parameters, as well as the model state, to match the true system more closely. Practical methods have previously been suggested to augment the dynamical equations for a pair of synchronously coupled systems with parameter adaptation equations in particular cases [10–12], but it has been argued that no general method exists [10]. We show here that a general method indeed exists for an important class of systems: those given by PDE's with constant coefficients, the general form for physical theories with translational symmetry.

To motivate the analysis of PDE systems, we first consider a simple "real system" given by ODE's:

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{p}), \tag{1}$$

$$\dot{\boldsymbol{p}} = 0, \tag{2}$$

where $\boldsymbol{x} \in \mathbb{R}^N$, $\boldsymbol{f} : \mathbb{R}^N \to \mathbb{R}^N$, and $\boldsymbol{p} \in \mathbb{R}^m$ is the vector of (unknown, constant) parameters of the system. We further assume that $\mathbf{s} = \boldsymbol{h}(\boldsymbol{x})$, where $\boldsymbol{h} : \mathbb{R}^N \to \mathbb{R}^n$, $n \leqslant N$, is an n-dimensional vector representing the experimental measurement output of the

* Corresponding author. Tel.: +1 303 497 1713; fax: +1 303 497 1700.
E-mail address: gduane@ucar.edu (G.S. Duane).

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system. A "computational model" of the system (1) is given by:

$$\dot{\boldsymbol{y}} = \boldsymbol{f}(\boldsymbol{y}, \boldsymbol{q}) + \boldsymbol{u}(\boldsymbol{y}, \mathbf{s}), \tag{3}$$

$$\dot{\boldsymbol{q}} = \boldsymbol{N}(\boldsymbol{y}, \boldsymbol{x} - \boldsymbol{y}), \tag{4}$$

where $\boldsymbol{N}(\boldsymbol{y}, 0) = 0$, and $\boldsymbol{u}$ is the control signal. Generally, the real system (1) and its model (3) are chaotic; for $\boldsymbol{u} = 0$ the simulation quickly diverges from the real system behavior.

In this Letter we address the design of a parameter estimation law $\boldsymbol{N}$, so that $\boldsymbol{q} \to \boldsymbol{p}$, along with a control law $\boldsymbol{u}$ that gives $\boldsymbol{y} \to \boldsymbol{x}$. Note that since $\boldsymbol{p} \neq \boldsymbol{q}$ the real system (1) and its model (3) never identically synchronize. However, as we show below, the system (1)–(2) may identically synchronize with (3)–(4). We first prove the following statement: Assuming that the systems (1) and (3) synchronize when $\boldsymbol{p} = \boldsymbol{q}$ for some control law $\boldsymbol{u}$, then under some mild mathematical conditions, one can always design $\boldsymbol{N}$ such that the system (1)–(2) and the system (3)–(4) identically synchronize.

The method of proof, following [10], is to use the monotonically decreasing Lyapunov function that must exist for the synchronously coupled identical systems to construct a monotonically decreasing Lyapunov function for the coupled nonidentical systems with one set of dynamically varying parameters. Let $\mathbf{e} \equiv \boldsymbol{y} - \boldsymbol{x}$ and $\mathbf{r} \equiv \boldsymbol{q} - \boldsymbol{p}$. Then consider the synchronization error system:

$$\dot{\mathbf{e}} = \boldsymbol{h}(\mathbf{e}, \boldsymbol{y}, \boldsymbol{p}, \mathbf{r}) + \boldsymbol{u}(\mathbf{s}, \boldsymbol{y}), \tag{5}$$

$$\dot{\mathbf{r}} = \boldsymbol{N}(\boldsymbol{y}, \mathbf{e}), \tag{6}$$

where $\boldsymbol{h} \equiv \boldsymbol{f}(\boldsymbol{y}, \mathbf{r} + \boldsymbol{p}) - \boldsymbol{f}(\boldsymbol{y} - \mathbf{e}, \boldsymbol{p})$. Since necessarily $\boldsymbol{N}(\boldsymbol{y}, 0) = 0$, for chaotic systems $\mathbf{e} \to 0$ in (5)–(6) implies also $\mathbf{r} \to 0$, as shown in [12].

Now consider the synchronization error system (5)–(6) for the case of identical parameters:

$$\dot{\mathbf{e}} = \boldsymbol{h}(\mathbf{e}, \boldsymbol{y}, \boldsymbol{p}, 0) + \boldsymbol{u}(\mathbf{s}, \boldsymbol{y}). \tag{7}$$

For the system (7), we choose a positive definite Lyapunov function $L_o(\mathbf{e})|_{\boldsymbol{q}=\boldsymbol{p}}$. Assume that the control signal $\boldsymbol{u}$ is designed such that there is some time t_0 for which $\dot{L}_o(\mathbf{e}(t))|_{\boldsymbol{q}=\boldsymbol{p}} < 0$ when $\mathbf{e}(t) \neq 0$ and $\dot{L}_o(\mathbf{e}(t))|_{\boldsymbol{q}=\boldsymbol{p}} = 0$ when $\mathbf{e}(t) = 0$, for all $t > t_0$. The time t_0 is a time by which all bursting has stopped and after which the systems are assumed to proceed monotonically toward synchronization, as they will for the many ODE and PDE configurations that have been found to exhibit "high-quality" synchronization.[1]

For the system (5), with differing parameters, we choose a Lyapunov function

$$L(\mathbf{e}, \mathbf{r}) = L_o(\mathbf{e})|_{\boldsymbol{q}=\boldsymbol{p}} - \int \sum_{i,j} \left(\frac{\partial L_o}{\partial e_i}\right)\left(\frac{\partial h_i}{\partial r_j}\right) r_j \, dt. \tag{8}$$

First assume that $\boldsymbol{f}$ is linear in the parameters $\boldsymbol{p}$, so that $\boldsymbol{h}$ is also linear in $\mathbf{r}$. It is easy to see that $\dot{L}_o(\mathbf{e}) = \dot{L}_o(\mathbf{e})|_{\boldsymbol{q}=\boldsymbol{p}} + \sum_{i,j} (\partial L_o/\partial e_i)(\partial h_i/\partial r_j) r_j$, so that $\dot{L}(\mathbf{e}, \mathbf{r}) = \dot{L}_o(\mathbf{e})|_{\boldsymbol{q}=\boldsymbol{p}}$. Choosing

$$\dot{r}_j = N_j \equiv -\delta_j \sum_i \left(\frac{\partial L_o}{\partial e_i}\right)\left(\frac{\partial h_i}{\partial r_j}\right),$$

where δ_j are positive constants, we have

$$L(\mathbf{e}, \mathbf{r}) = L_o(\mathbf{e})|_{\boldsymbol{q}=\boldsymbol{p}} + \int \sum_j \frac{\dot{r}_j}{\delta_j} r_j \, dt = L_o(\mathbf{e})|_{\boldsymbol{q}=\boldsymbol{p}} + \sum_j \frac{r_j^2}{2\delta_j}. \tag{9}$$

We conclude that L is a positive definite Lyapunov function such that $\dot{L} < 0$ when $\mathbf{e} \neq 0$.

Therefore, the following theorem has been proved.

Theorem 1. *Assume that*

(i) *the control law $\boldsymbol{u}$ in (7) is designed such that the synchronization manifold $\boldsymbol{x} = \boldsymbol{y}$ is globally asymptotically stable,*
(ii) *$\boldsymbol{f}$ is linear in the parameters $\boldsymbol{p}$, and*
(iii) *the parameter estimation law (4) is designed such that*

$$N_j = -\delta_j \sum_i \left(\frac{\partial L_o}{\partial e_i}\right)\left(\frac{\partial h_i}{\partial r_j}\right),$$

where δ_j are positive constants. Then the synchronization manifold $\boldsymbol{y} = \boldsymbol{x}$, $\boldsymbol{p} = \boldsymbol{q}$ is globally asymptotically stable.

The main assumption of Theorem 1, that one can design the control law $\boldsymbol{u}$ such that the systems (1) and (3) with $\boldsymbol{p} = \boldsymbol{q}$ globally synchronize, in practice cannot always be easily satisfied. However, *local* stability of the synchronization manifold $\boldsymbol{x} = \boldsymbol{y}$ (for identical synchronization) is more easily achieved. The theorem and its proof are readily modified for the case of local stability of the identical synchronization manifold.

The premise of linearity in the parameters is not very restrictive. For instance, if $\boldsymbol{f}$ is a polynomial function of the parameters $\boldsymbol{p}$, one can regard each of the products $p_{i1} p_{i2} \ldots$ that appears in the expansion as a separate parameter, on which $\boldsymbol{f}$ depends linearly. An application of the parameter update rule to the new parameters will cause the parameters of the model system corresponding to the quantities $q_{i1} q_{i2} \ldots$ to converge to the desired values. Other re-parameterizations can be used for more general functional forms $\boldsymbol{f}$. One is challenged to find a physically relevant system of differential equations for which a change of variables, together with the introduction of extra parameters, will not yield a system that meets the requirement of linearity in parameters.

The theorem ensures the stability of the synchronization manifold $\boldsymbol{y} = \boldsymbol{x}$, $\boldsymbol{p} = \boldsymbol{q}$. It says: identical synchronization implies parameter estimation, provided that each partial derivative $\partial L_o/\partial e_i$ is known for which the vector $\partial h_i/\partial r_j$ $(j = 1, \ldots)$ is not zero. For the usual form $L_o \equiv \sum_i (e_i)^2$, the requirement is that $\boldsymbol{x}_i$ be known if the equation for $\dot{\boldsymbol{y}}_i$ contains parameters that one seeks to estimate. By considering a more general Lyapunov function that is defined in terms of some subset S of the state

[1] Formally, the requirement for high-quality synchronization is that not only are conditional Lyapunov exponents negative for chaotic trajectories, but they are also negative for all unstable periodic orbits embedded within the attractor.

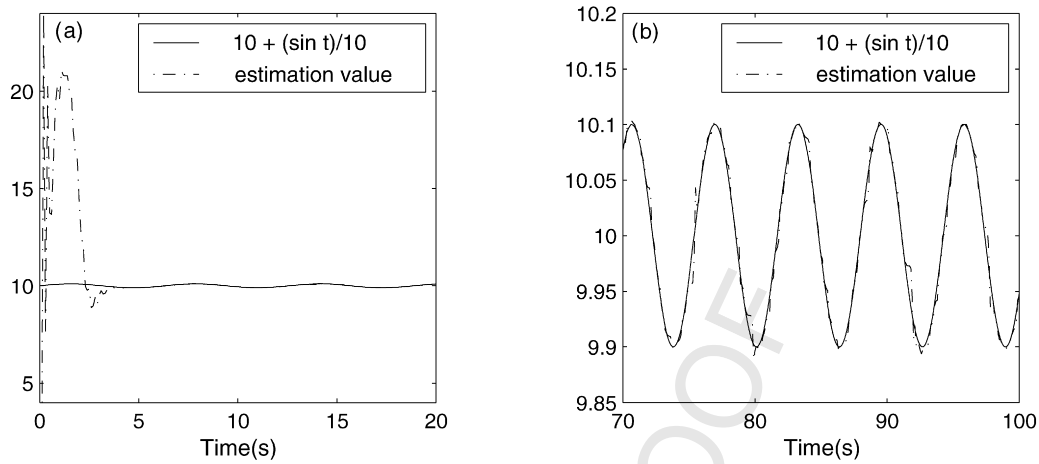

Fig. 1. The Lorenz parameter "σ" vs. time for a 'real' system with slowly varying $\sigma = p = 10 + \frac{1}{10}\sin t$, $\rho = 28$ and $b = 8/3$; and for a 'model' with $\rho = 28$, $b = 8/3$ and σ estimated. As $t \to \infty$ the model parameter q estimates the parameter p of the real system (a). Results are also shown at increased resolution in σ after synchronization (b).

variables, or their indices, $L_o \equiv \sum_{i \in S} c_i(e_i)^2$ for positive coefficients c_i, one obtains the looser requirement for each desired parameter, that $\boldsymbol{x}_i$ be known for at least some i for which the $\dot{\boldsymbol{y}}_i$ equation contains that parameter. (Convergence may be slower if fewer $\boldsymbol{x}_i$ are known.)

As a specific example of parameter estimation, consider the Lorenz system $\dot{x}_1 = p(x_2 - x_1)$, $\dot{x}_2 = \rho x_1 - x_1 x_3 - x_2$, and $\dot{x}_3 = x_1 x_2 - b x_3$, where p is an unknown parameter (while ρ and b are known parameters). Let $s = x_1$ describe a measurement (or observation) variable. The "model" is a second Lorenz system $\dot{y}_1 = q(y_2 - y_1) + u$, $\dot{y}_2 = \rho y_1 - y_1 y_3 - y_2$, $\dot{y}_3 = y_1 y_2 - b y_3$, and $\dot{q} = N(\boldsymbol{y}, \mathbf{e})$. Let the Lyapunov function $L_o(\mathbf{e}) \equiv e_1^2 + e_2^2 + e_3^2$ and choose $u = -(k_1 + k_2 y_2^2 + y_3^2/2)e_1$. For $k_1 > -p + (p + \rho)^2/2$ and $k_2 > 1/(4b)$, it is readily shown that $\dot{L}_o < 0$ globally if $\mathbf{e} \neq 0$. In this case, Theorem 1 gives the parameter estimation rule:

$$\dot{q} = \dot{r} = N = a(y_1 - y_2)e_1 \tag{10}$$

for arbitrary positive constant a. Under the rule (10) the synchronization manifold $\boldsymbol{x} = \boldsymbol{y}$, $p = q$ is globally asymptotically stable.

The method given by Theorem 1 succeeds even when the parameter depends on time and when the system has noise. Consider a Lorenz system for which the parameter p of the real system varies with time as $p = 10 + \frac{1}{10}\sin t$. Again the model correctly estimates the parameter. Fig. 1 illustrates this example with $N(y, e) = a(y_1 - y_2)e_1$ and $a = 100$. As will be reported elsewhere, noise is tolerated at a level that does not take the system out of the basin of attraction of the synchronization manifold, provided that high-quality synchronization obtains.

In general, some of the partial derivatives $\partial L_o/\partial e_i$ may not be known and Theorem 1 may be inapplicable. The theorem gives us no way to estimate the parameters ρ or b for the Lorenz system, for instance, since only x_1 is coupled, while x_2 and x_3 (hence $\partial L_o/\partial e_2$ and $\partial L_o/\partial e_3$) are unknown. But in the case of translationally invariant PDE's, the parameters are the same at each point in space. In general, they can be estimated from a limited amount of information about the state at a discrete set of points (or for a finite set of Fourier components), if such information is also sufficient to give identical synchronization globally or locally when coupled to a "model system". Consider a pair of spatially extended systems that are synchronously coupled in one direction:

$$\begin{aligned}
&\frac{\partial \phi^A}{\partial t} + \Gamma(\phi^A) = f^A F(\phi^A, \phi^A), \\
&\frac{\partial \phi^B}{\partial t} + \Gamma(\phi^B) + C(\phi^A, \phi^B) = f^B F(\phi^B, \phi^A),
\end{aligned} \tag{11}$$

where Γ is some general form in the field ϕ, possibly involving spatial derivatives, C is a similarly general form coupling ϕ^B to ϕ^A, such that $C(\phi, \phi) = 0$, and the form F with coefficient f specifies a "forcing", which in general may also couple to the other system. The two systems are dynamically identical when $\phi^A = \phi^B$ and $f^A = f^B$.

Define a core Lyapunov function $L_o(\phi^A - \phi^B)|_{f^A = f^B} \equiv \int d^3x\,(\phi^A - \phi^B)^2$ and choose a Lyapunov function for the general case of unequal parameters:

$$L \equiv (f^A - f^B)^2 + \int d^3x\,(\phi^A - \phi^B)^2 \tag{12}$$

corresponding to the form (9) for the ODE case.

Assume that L_o is monotonically decreasing, say

$$\frac{1}{2}\dot{L}_o(f^A, f^A) = \mathcal{N}(\phi^A, \phi^B)$$

where $\mathcal{N}$ is some negative semi-definite form, generally not quadratic, in ϕ^A and ϕ^B. Then a ready generalization of Theorem 1, and its proof, to the case of a continuum of state variables tells us that if we impose the parameter estimation law

$$\dot{f}^B = \int d^3x\,(\phi^A - \phi^B) F(\phi^B, \phi^A), \tag{13}$$

we will find $\frac{1}{2}\dot{L} = \mathcal{N}$ and so the time-derivative of the Lyapunov function for unequal parameters is also negative semi-definite. Since the expression (12) for L is also bounded below,

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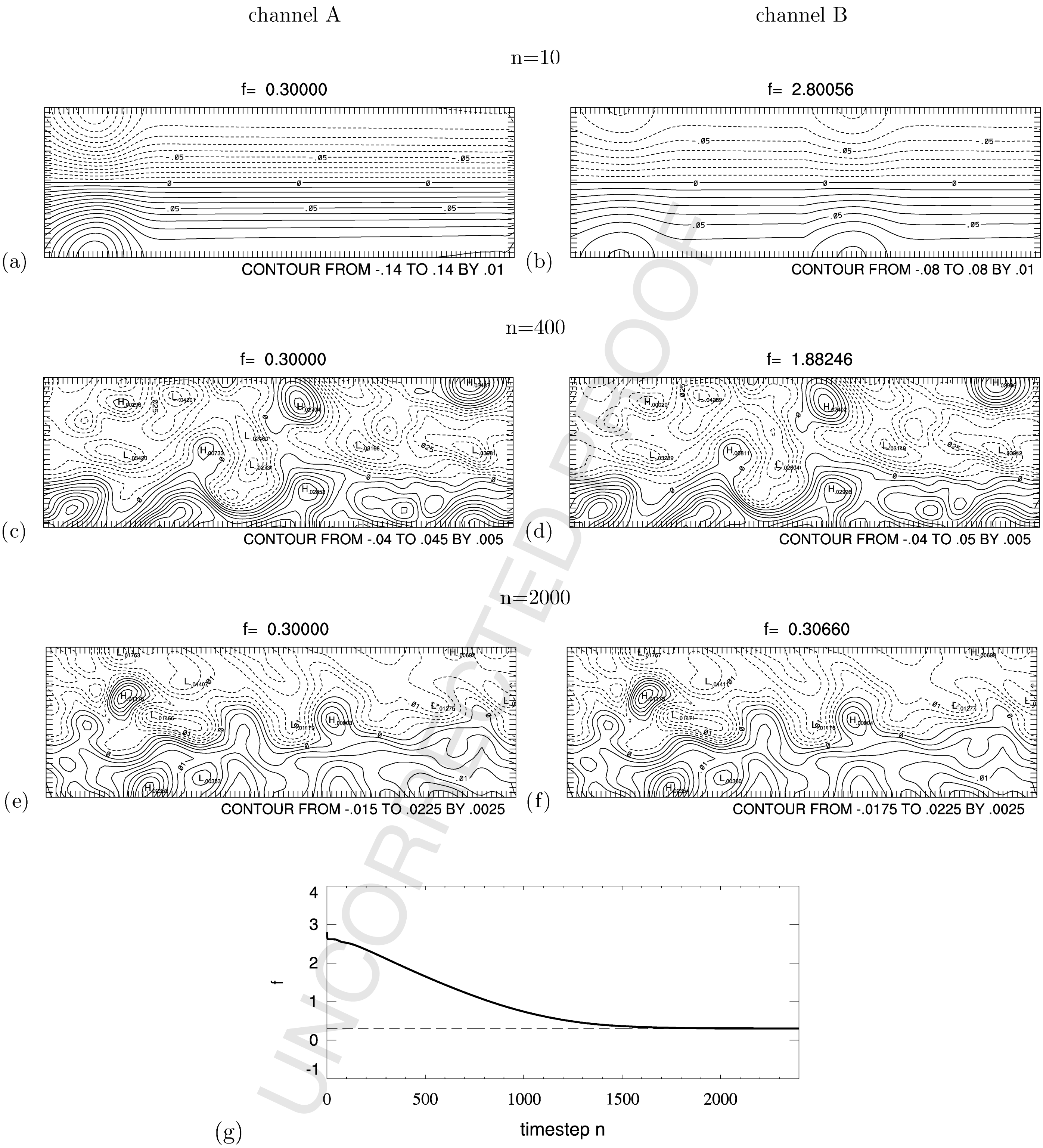

Fig. 2. The evolving flow ψ (a)–(f) for two quasigeostrophic channel models that are synchronously coupled as in [4,5] (but in one direction only), and with the forcing parameter f^B for the second channel (denoted μ_o in the reference) allowed to vary according to the truncated parameter adaptation rule (16) with $S = \{\mathbf{k}\colon \mathbf{k}_x, \mathbf{k}_y \leqslant 12\}$ (in waves per-channel length). Starting from the initial value $f^B = 3.0$ at time step $n = 0$ (not shown), f^B converges (g) to the value of the corresponding parameter $f^A = 0.3$ (dashed line) in the first channel, as the flows synchronize. (An average of the two layers $l = 1, 2$ is shown.)

we have $f^B \to f^A$ as in the ODE case, provided that spurious zeros of $\mathcal{N}$ can be avoided.

The procedure outlined above estimates a coefficient of forcing for a wide class of synchronously coupled PDE's, and is readily generalized for limited measurements of ϕ^A. The quasigeostrophic potential vorticity equation, for instance, used to describe the large-scale atmospheric circulation, has been shown to exhibit high-quality synchronization when two copies

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are coupled via partial exchange of only mid-range Fourier components of the flow field [4,5]. The synchronization manifold is either globally attracting or locally attracting in a very wide basin. Synchronization of such systems may be useful for meteorological data assimilation [8,13]. It is important that forcing parameters in the equation can also be estimated. The equation for potential vorticity q in a two-layer channel is

$$\frac{Dq_l}{Dt} \equiv \frac{\partial q_l}{\partial t} + J(\psi_l, q_l) = F_l + D_l, \tag{14}$$

where the vertical layer $l = 1, 2$, $\psi_l(x, y)$ is streamfunction, F is a forcing term, D is a dissipation term, the Jacobian $J(\psi, \cdot) = \frac{\partial \psi}{\partial x}\frac{\partial \cdot}{\partial y} - \frac{\partial \psi}{\partial y}\frac{\partial \cdot}{\partial x}$ gives the advective contribution to the co-moving ("Lagrangian") derivative D/Dt, and the potential vorticity field, derived from the streamfunction, is given by

$$q_l = f_0 + \beta y + \nabla^2 \psi_l + R_l^{-2}(\psi_1 - \psi_2)(-1)^l,$$

where the coefficients are defined in [5]. For a forcing of the form $F_l = f(q_l^* - q_l)$, where the field q_l^* represents a fixed pattern to which the flow is forced to relax, one might seek to estimate the coefficient f.

Consider two systems, "A" and "B", both given by equations of the form (14), but with the forcing in the B system defined differently, in terms of its spectral components $F_{\mathbf{k}}^B$, so as to effect a unidirectional coupling:

$$F_{\mathbf{k}}^B = f^B \sum_{\mathbf{k}} a_{\mathbf{k}}(q_{\mathbf{k}}^* - q_{\mathbf{k}}^B) + f^B \sum_{\mathbf{k}} b_{\mathbf{k}}(q_{\mathbf{k}}^A - q_{\mathbf{k}}^B), \tag{15}$$

where the layer index l is suppressed and the coefficients $a_{\mathbf{k}}, b_{\mathbf{k}}$ are slightly smoothed step functions of $\mathbf{k}$, so that each spectral component is either coupled to the corresponding component in the A system or to the background flow q^* or neither. The coefficients are chosen so as to couple only the medium-scale components:

$$b_{\mathbf{k}} = \begin{cases} 0 & \text{if } |k_x| \leqslant k_{x0} \text{ and } |k_y| \leqslant k_{y0}, \\ (k_n/|\mathbf{k}|)^4 & \text{if } |\mathbf{k}| > k_n, \\ 1 - (k_0/|\mathbf{k}|)^4 & \text{otherwise} \end{cases}$$

and to force only the large-scale components:

$$a_{\mathbf{k}} = \begin{cases} 1 - b_{\mathbf{k}} & \text{if } |\mathbf{k}| \leqslant k_n, \\ 0 & \text{if } |\mathbf{k}| > k_n, \end{cases}$$

as in [5], where the constants k_0, k_{x0}, k_{y0} and k_n are defined. The systems thus coupled synchronize, without bursting, as shown in Fig. 2. (The forcing for the A system is correspondingly truncated: $F_{\mathbf{k}}^A = f^A \sum_{\mathbf{k}} a_{\mathbf{k}}(q_{\mathbf{k}}^* - q_{\mathbf{k}}^A)$.)

The parameter estimation rule in spectral space:

$$\dot{f}^B = \sum_{\mathbf{k} \in S} (q_{\mathbf{k}}^A - q_{\mathbf{k}}^B)\left[a_{\mathbf{k}}(q_{\mathbf{k}}^* - q_{\mathbf{k}}^B) + b_{\mathbf{k}}(q_{\mathbf{k}}^A - q_{\mathbf{k}}^B)\right], \tag{16}$$

is the Fourier transform of (13) if S is universal. But even for a restricted range of wavenumbers in S, as in the figure, the rule (16) causes f^B to converge to f^A as would follow from the use of a correspondingly restricted Lyapunov function.

Analogously, suppose that the coupling C that gives identical synchronization in (11) for $f^A = f^B$ is defined in terms of a discrete set of points, i.e., $C(\phi^A(x), \phi^B(x)) = 0$, $F(\phi^B(x), \phi^A(x)) = F(\phi^B(x), \phi^B(x))$ unless $x \in S$, where S is a discrete set of points, as in [3]. Then the use of a correspondingly restricted Lyapunov function $L_o \equiv \sum_{x \in S}[\phi_A(x) - \phi_B(x)]^2$ implies that the integral in (13) is effectively estimated by the sum $\sum_{x \in S}(\phi^A(x) - \phi^B(x))F(\phi^B(x), \phi^A(x))$.

The rule (13) states that the coefficient of forcing is to be increased or decreased as the covariance of the local forcing and the local error. If the spatial integral in (13) is replaced by a time average, then the resulting rule for varying the coefficient is analogous to the Hebbian learning rule, according to which the strength of synaptic coupling is increased or decreased as the covariance between pre- and post-synaptic potentials.

In conclusion, we have proved the following: If a translationally symmetric physical system and a model with identical parameters synchronize and some parameters of the real system are unknown, but some state variables whose equations contain those parameters are observed at the discrete set of points (or for the finite set of Fourier components) that are coupled to the synchronizing model, then one can design a parameter estimation law so that all parameters of the real system can be estimated. The result can be applied to adapt any PDE model of a physical system based on a limited set of ongoing observations.

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